Selected Passages from Correspondence with Friends 14 no. 2 (2026), 5–13. ISSN 2054–7145

# YIN AND YANG OF ELEMENTARY GEOMETRY

ALEXANDRE BOROVIK

A pre-publication version of a paper submitted for publication.

Abstract

Triangles and many other geometric figures in the plane could be *chiral*, not superimposable on their mirror symmetry images. This paper is a sketch of a rigorous treatment of the concepts of chirality and orientation which could be used in more deep or advanced streams of school level mathematics. It is based on treating the Euclidean plane as

> A 2-dimensional real vector space with two non-degenerate bilinear forms on it, one symmetric and positive definite and another—antisymmetric (symplectic).

These two bilinear forms are the eponymous Yin and Yang of the title of the paper.

Most proofs are omitted – they are fairly obvious. The paper contains some historic comments and my personal stories, quite relevant to this topic.

## 1. *Introduction*

1.1. *The intended readers.* The paper is written for mathematicians and mathematical educationalists who think about and work on new models of mathematics education fit for the world of the 21st century.

1.2. *The new setup of mathematics education.* The present paper is motivated by my attempts to develop an*AI-independent* synthetic approach to teaching school arithmetic, algebra, and computer programming as a single subject in school mathematics [3]. It should be taught under a new paradigm of mathematics education:

> Starting from primary school, solution of a “computational” problem should be not only a number, or “yes/no”, or an algebraic expression, but an algorithm or computer code which solves *all* problems of the same type.

I appreciate that this is s strong statement. My paper [3] contains a detailed discussion and a proposal for a new unified course *Arithmetic – Algorithmic – Algebra* which includes a serious exposure to computer programming and covers all school years. This is a course for future *masters, not slaves, of AI.*

This synthetic course requires a *domain specific language* (DSL) [1, 8, 9] which would allow Child to talk about mathematics with Computer. Moreover, this course will require a more friendly, than usual, user computer interface—it has to be *child friendly.* This is a very serious challenge.

The DSL also requires a much more formal mathematical language, mostly hidden from the user, but with fragments accessible to Child and Teacher. This sets

---

2020 *Mathematics Subject Classification* 97G60 (primary), 97H60 (secondary).

an equally important task of a deep *didactic transformation* of the teaching material, development of completely new mathematical approaches to the content of the courses.

Jean Dieudonné said in the Introduction to his book *Algèbre linéaire et géométrie élémentaire* [6]

> I hope that I will be believed if I note at the end that I do not pursue any personal gain by interfering in matters of secondary education. Where and when the education reform will take place and what principles will be laid at its foundation are neither hot nor cold to me. I only wanted to add some material for the archives of the future historian about how one could proceed in this regard if one strives to act wisely.

Like Dieudonné, I can claim that I 'do not pursue any personal gain' by writing this paper, but unlike him, I take the fate of mathematics education and health of our mathematical community pretty close to heart.

1.3. *Algebraisation of geometry with preservation of the geometric content.* The present paper contains a discussion of a small but illuminating fragment of the *Arithmetic – Algorithmic – Algebra* proto-curriculum. I propose that elementary geometry, treated as vector algebra in dimensions 2 (and perhaps 3) should be part of school algebra. In that case, the issue of *orientation* immediately raises its ugly head.

> **Example 1.** How to explain, to Child and to Computer, and how Child will be later talking to Computer, in a language understood to, and *actionable* by, both of them, the meaning of the phrase:
>
> The vertices $A$, $B$, $C$ of the triangle $\triangle ABC$ are listed in the counterclockwise order.

Indeed, what does it mean, *counterclockwise*? How to enter this information in a computer?

Or another problem:

> **Example 2.** The points $A$, $B$, $C$, and $D$ are given by their Cartesian coordinates on the plane. Determine whether the point $D$ lies inside of the triangle $\triangle ABC$, or not.

In the case of elementary geometry, it was already known for quite a while that the most efficient way to handle chirality and orientation in the Euclidean plane geometry is to axiomatise the latter as

> A 2-dimensional real vector space with two non-degenerate bilinear forms on it, one symmetric and positive definite and another—antisymmetric (symplectic).

See, for example, [6]. The two bilinear forms could be seen as *Yin and Yang* of Euclidean geometry, Fig. 1. From Wikipedia:

> Yin and yang can be thought of as complementary and at the same time opposing forces that interact to form a dynamic system in which the whole is greater than the assembled parts and the parts are important for cohesion of the whole.

The bilinear symmetric positive definite form is well known under the name of the *scalar product* or *dot product*. The symplectic one (I call it in the present paper the *wedge product*) inhabits mostly advanced chapters of mathematics.

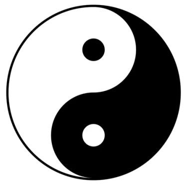

FIGURE 1. *Yin and Yang.*

1.4. *Chirality and orientation in the real world.* In the physical Universe, at every scale—from astronomical to molecular and subatomic, orientation manifests itself in ways critically important for us, humans.

Imagine that you look at the Solar system from some point in between the Sun and Polaris (also known as North Pole Star). You see a yellow blot—Sun, very slowly orbited around by a smaller blue blot—Earth. In which direction: clockwise or counterclockwise? Effects of that can be frequently found in everyday life.

OK, a simpler, but even more consequential (the answer affects the behavior of tides on the seas) question: does the Moon rotates around Earth in the same direction as Earth rotates around the Sun?

Or even closer to everyday life: should we care about the direction of thread on bolts matching the direction of thread on nuts? How did humanity manage to reach standardisation of threads on bolts and nuts? Apparently, because 95% of people have the right hand as dominant—but following this line of enquiry we eventually find ourselves in the realm of molecular biology, where *chiralty* of many molecules, that is, impossibility to superimpose them on their mirror images by any combination of translations or rotations, plays quite an important role. Orientation of chiral molecules in living organisms is strictly controlled, for the same reason why direction of thread on nuts and bolts is standardised in the industry.

Orientation was, and remains, the most misunderstood and neglected area of elementary geometry. Here, I am trying to explain it. The main difficulty is that it belongs to *symplectic geometry*, that is, the geometry of an anti-symmetric bilinear form, not to traditional Euclidean geometry as it is taught (or was taught) in schools. Indeed, when we have in the plane an orthonormal basis $\overrightarrow{E}_1$, $\overrightarrow{E}_2$ with respect to a positively determined symmetric bilinear form, swapping $\overrightarrow{E}_1$ and $\overrightarrow{E}_2$ extends to an automorphism of the Euclidean geometry on the plane—but this automorphism changes the orientation of chiral figures in the plane. Perhaps the simplest of them is a *scalene* (that is, non-isosceles) triangle, and it is discussed in Section 2.

In this paper, I always keep in mind the interaction between Child and Computer; mathematics that appears to be sophisticated does not need to be necessarily revealed to children; perhaps it would be best to hide it under the hood of the computer system. But the need for developing of a language shared by Teacher, Child, and Computer remains.

## 2. *Chirality of a scalene triangle*

In this section I temporary switch to telling my personal story. I hope the reader will excuse me for deviating from the strictly academic style of writing and will find the story relevant.

2.1. *Holes and chirality.* This is a problem which intrigued me my whole life.

> **Problem.** A scalene triangle (that is, a triangle in which all three sides are in different lengths) is drawn on a piece of paper. It was carefully cut out and turned over. Prove that if you move the flipped over triangle along a sheet of paper, then it cannot be used to cover the hole left behind without any gaps.

This problem was given to me by Valentin Nikolaevich Monakhov, then Head of Mathematics Department of the Novosibirsk State University (NSU), at the admission interview to the Physics and Mathematics School in August 1971. I was 15 years old but had read a couple of books[†] and timidly told Monakhov that this could not be proven within the framework of school geometry, because the latter did not have the necessary concepts. My answer produced an amazing effect: Monakhov jumped up from his chair and began jumping up and down, slapping his thighs with his palms, and laughing. Having calmed down, he told me that I had been accepted to the PhMSh, and let me go. Luckily, he did not ask me to develop my answer in more detail, most likely I would not be able to do that. I can do it now, 50 years later.

Much later my friends explained to me that Monakhov was known for being exuberant in expression of his emotions, and even later I learned that at that very time (1970–1973) Monakhov was the Chairman of the Scientific Council of the Siberian Branch of the USSR Academy of Sciences on educational issues and, in particular, had to deal with Kolmogorov's reform of mathematics in school, where geometry turned out to be even more hopelessly confused [2].

The problem is about the concept of plane orientation, which, as far as I know, is not taught in school in any country in the world.

2.2. *Flatland and Möbius strip.* The concept of orientation could be conveniently explained using two examples of flat two-dimensional worlds. One of them is *Flatland*[‡]. I never read the book, it was published in Russia only in 1976, when I was already fully engaged in mathematical research. But somewhen in 1970–71 I read an article in *Quantum* magazine, about Flatland, which took care to explain that all men in Flatland faced one way and women the other, "so that they could kiss," and even had a picture: if a man wanted to kiss his son, he had to hold him upside down above his head.

It is very instructive to compare Flatland with the Möbius strip. Locally, on a small piece, the Möbius strip is no different from a piece of the ordinary Euclidean plane, that is, Flatland. You can experiment with a scalene triangle on the Möbius strip, cut out and flipped over: if you drag it near the hole, you can't cover the hole. But if you drag it along the entire length of the strip, it covers it perfectly.

The same thing happens with the orientation of people living on the Möbius strip. Having traveled around the world, they change their orientation.

The moral of the story: there are surfaces that are locally indistinguishable from the Euclidean plane; among them, the plane itself, the cylinder, the torus (with a

---

[†]In the year before this interview, I have read, among other books, Russian translations of Coxeter's *Introduction to Geometry* [5] and Robin Hartshorn's *Foundations of Projective Geometry* [7]; in the latter, I was mesmerised by the level of abstraction and strictly axiomatic approach to the development of the theory and fascinated by the role of groups and group actions.

[‡]From Wikipedia: "*Flatland: A Romance of Many Dimensions* is a novel by Edwin Abbott Abbott that was published in 1884."

modified metric on it) are orientable, that is, they have the notion of orientation, but the Möbius strip and the Klein bottle are not [10][§]. Orientability is a *global* property of a surface, not a *local* one. Traditional school geometry, as it was taught for centuries, was purely local.

When I talked to Monakhov, I did not know all that stuff yet, but I had read something and felt that one must be very careful with orientation.

2.3. *Cosmology.* I was privileged to work with Israel Gelfand, one of the great mathematicians of the 20th century. I did some research in combinatorics with him, in particular, the combinatorics of maps on different surfaces (like geographical maps on a globe). It was when I started to think about maps on non-orientable surfaces, that the metaphor of travelling around the Moebius strip crossed my mind.

And the possibility of a kiss between two men of different orientations moved my imagination further: what if our 4-dimensional space-time is a non-orientable manifold and an electron, having traveled around the Universe, comes back as a positron (i.e. anti-electron), and annihilates upon meeting its fellows, and the same happens with all elementary particles? This is the reason why locally, around us, there is no antimatter. Antiparticles can be generated, under special conditions, as part of a pair (particle — antiparticle), in a process reverse to annihilation, spending a lot of energy on this, but antiparticles do not live long, only until their first collision with matter.

I came to a friend, a theoretical physicist, with my bold cosmological hypothesis. Having heard my story, he visibly changed. His face turned black; he grabbed the table with his hands and sat that way for a minute or two in a daze. Then he looked up at me and said: "You know, Sasha, I wasted two years of my life trying to develop this theory."

2.4. *A bit about Artificial Intelligence.* Attempts to ask Monakhov's questions to LLM systems, COPILOT and JULIUS, produced bizarre results. Perhaps this means that the Internet, their feeding ground, does not contain quality material explaining chirality and orientation. These concepts are absent from the mass mathematical culture.

## 3. *Vectors*

I will be using some elementary vector algebra in 2 dimensions normally taught in secondary schools: vectors, bases, coordinates, equations for straight lines.[†]

Pick the origin $O$ of a coordinate system in the plane; every vector is equal to a vector of the form $\overrightarrow{OX}$, and every point $X$ in the plane gives rise to its point vector $\overrightarrow{OX}$.

Now pick two coordinate vectors $\overrightarrow{E}_1$ and $\overrightarrow{E}_2$.

[§]More precisely, the book deals with the *twisted cylinder*, that is, the Möbius strip of infinite width, so that it has no boundary.

[†]A proper exposition of that material would benefit from a consistent and accessible explanation of *free* and *bound* vectors and, in effect, a gentle introduction of an *affine plane* in sense of Hermann Weyl. This terminology is, of course, not for kids. Vectors are a delicate matter, the great Andrei Kolmogorov famously stumbled on them in his course of school geometry.

Any vector $\overrightarrow{U}$ can be expressed as a linear combination

$$\overrightarrow{U} = u_1\overrightarrow{E}_1 + u_2\overrightarrow{E}_2$$

of the coordinate vectors. We ask a very natural question: are vectors

$$\overrightarrow{U} = u_1\overrightarrow{E}_1 + u_2\overrightarrow{E}_2 \quad \text{and} \quad \overrightarrow{V} = v_1\overrightarrow{E}_1 + v_2\overrightarrow{E}_2$$

parallel? (in this situation, it is the same as collinear).

## 4. *Antisymmetric bilinear form*

The further development is done on the assumption that it is obvious for the reader that if vectors $\overrightarrow{U}$ and $\overrightarrow{V}$ are collinear than there exists a scalar $\lambda$ such that one of the vectors, say $\overrightarrow{U}$, is obtained from $\overrightarrow{V}$ by a multiplication by $\lambda$, or vice versa,

$$\overrightarrow{U} = \lambda\overrightarrow{V} \text{ or } \lambda\overrightarrow{U} = \overrightarrow{V}$$

which is equivalent to

$$\left\{ \begin{array}{lcl} u_1 & = & \lambda v_1 \\ u_2 & = & \lambda v_2 \end{array} \right. \text{ or } \left\{ \begin{array}{lcl} \lambda u_1 & = & v_1 \\ \lambda u_2 & = & v_2 \end{array} \right. ,$$

which could be easily shown to be equivalent to

$$u_1v_2 - u_2v_1 = 0.$$

Denote

$$\overrightarrow{U} \wedge \overrightarrow{V} = u_1v_2 - u_2v_1.$$

So the criterion for the vectors $\overrightarrow{U}$ and $\overrightarrow{V}$ to be collinear is

$$\overrightarrow{U} \wedge \overrightarrow{V} = 0.$$

It is easy to check that $\overrightarrow{U} \wedge \overrightarrow{V}$ is a non-degenerate antisymmetric bilinear form. We shall call it the *wedge product.*[†] Of course, it is just the determinant of the $2 \times 2$ matrix formed from the coordinates of the vectors $\overrightarrow{U}$ and $\overrightarrow{V}$. But it is important that it is introduced naturally, and it has many handy applications—for example, it gives a very compact form for the equation of the line defined by two its points $A$ and $B$:

$$\overrightarrow{AX} \wedge \overrightarrow{AB} = 0.$$

## 5. *Orientation*

5.1. *Right and left.* So we have, in a fixed coordinate system $(\overrightarrow{E}_1, \overrightarrow{E}_2)$ on the plane, an antisymmetric bilinear form

$$\overrightarrow{U} \wedge \overrightarrow{V} = u_1v_2 - u_2v_1.$$

It is easy to see that it is not degenerate. Notice that

$$\overrightarrow{E}_1 \wedge \overrightarrow{E}_2 = 1$$

[†]For connoisseurs: what we have just done is known in model theory as *quantifier elimination*: we removed from the criterion of collinearity the quantifier $\exists\lambda$. Mathematics for children and computers has to be quantifier free.

and

$$\overrightarrow{E}_2 \wedge \overrightarrow{E}_1 = -1.$$

We shall say that an ordered pair of linearly independent (that is, non-collinear) vectors $(\overrightarrow{U}, \overrightarrow{V})$ has *the same orientation* as an ordered pair of linearly independent vectors $(\overrightarrow{W}, \overrightarrow{X})$ if and only if $\overrightarrow{U} \wedge \overrightarrow{V}$ and $\overrightarrow{W} \wedge \overrightarrow{X}$ have the same signs. The ordered pairs of vectors $(\overrightarrow{E}_1, \overrightarrow{E}_2)$ and $(\overrightarrow{E}_2, \overrightarrow{E}_1)$ have *opposite orientations*.

It is important to prove (this is easy, but I do not do that here) that this relation, *to have the same orientation*, does not depend on the choice of a basis in the plane.

Observe that a vector $\overrightarrow{V}$ defines the left and the right open half-planes as sets of vectors $\overrightarrow{U}$ such that $\overrightarrow{U} \wedge \overrightarrow{V} > 0$ and $\overrightarrow{U} \wedge \overrightarrow{V} < 0$, respectively. The corresponding closed half-planes are given by inequalities $\overrightarrow{U} \wedge \overrightarrow{V} \geqslant 0$ and $\overrightarrow{U} \wedge \overrightarrow{V} \leqslant 0$. It is easy to prove that half-planes are convex.

Another useful property: if points $X$ and $Y$, none of which belongs to the line $AB$, lie on the opposite sides of the line $AB$ if and only if the segment $[X, Y]$ intersects the line.

After that, we have a natural definition of the interior of a triangle $\triangle ABC$: it is a set of points $\overrightarrow{X}$ such that

$$\overrightarrow{AX} \wedge \overrightarrow{AB}, \quad \overrightarrow{BX} \wedge \overrightarrow{BC}, \quad \overrightarrow{CX} \wedge \overrightarrow{CA}$$

are all not equal 0 and have the same signs, that is, one can model a walk $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ along the triangle's sides and check whether the point $\overrightarrow{X}$ is always on its right or always on its left; if this indeed happens then $\overrightarrow{X}$ is inside the triangle. There is another walk around the triangle: $\overrightarrow{AC}$, $\overrightarrow{CB}$, $\overrightarrow{BA}$, but it bounds the same interior. The interior of a triangle is the intersection of three open half-planes and is therefore convex.

In fact, we get much more: if we look in the direction of vector $\overrightarrow{V}$, we have the concept of *left* and *right*. Also, we have the concept of counterclockwise and clockwise directions on the perimeter of a triangle (or of any convex polygon) or on a circle.

I leave it to the reader to answer

- Monakhov's question in this enriched setting.
- Similar questions on the cylinder and on the torus.

and analyse Examples 1 and 2 from Introduction.

And finally observe that all the stuff that we were discussing in this Section is easily convertible in computer code.

5.2. *Oriented area.* There is another important property of the wedge product: $\overrightarrow{U} \wedge \overrightarrow{V}$ can be interpreted as the oriented area of the parallelogram spanned by $\overrightarrow{U}$ and $\overrightarrow{V}$, with the expected properties of area:

- Additivity for areas of the same orientation.
- Invariance under parallel translations.
- After developing Euclidean geometry—invariance under rotations and change of sign under a mirror symmetry in a line.

Theory of oriented area can be developed with the use of the wegde only. The symplectic geometry is geometry of a *parallel ruler*, an instrument which allows to draw parallel lines in the same sense as Euclidean geometry is the theory of compasses and a straight-edge. We are allowed only three geometric constructions:

- Draw a line through two distinct points.
- Find the intersection of two lines, if it exists.
- For a given line $\ell$ and a point $P$, draw the line trough $P$ parallel to $\ell$.

Areas could be introduced and studied before metric. This was an approach chosen by Israel Gelfand in his geometry textbook he was developing in the late 1990s but did not have time to complete. Gelfand had a habit of involving his collaborators in all his project, and I did a bit of work helping him. Interestingly this experience helped me in my work on probabilistic algorithms for computational finite groups theory [4].

I do not know whether symplectic geometry deserved to be developed in school teaching beyond the chirality /orientation bit, where its use is inevitable. But it can provide a plethora of relatively simple problems for course projects etc..

## 6. *Complex numbers*

6.1. *The synthesis of Yin and Yang.* I have already quoted from Wikipedia:

> Yin and Yang can be thought of as complementary and at the same time opposing forces that interact to form a dynamic system in which the whole is greater than the assembled parts.

The two bilinear forms: symmetric and antisymmetric are present inside the field $\mathbb{C}$ of complex numbers seen as 2-dimensional real vector space. To emphasise the role of the real unity 1 and imaginary unity $i$, I will denote them using a boldfaced font: $\mathbf{1}$ and $\boldsymbol{i}$.

If

$$u = u_1\mathbf{1} + u_2\boldsymbol{i} \text{ and } v = v_1\mathbf{1} + v_2\boldsymbol{i}$$

are two complex numbers (where $u_1, u_2, v_1, v_2$ are real numbers) then their Hermitian product $\star$ is defined as

$$\begin{aligned} u \star v &= \bar{u}v \\ &= (u_1\mathbf{1} - u_2\boldsymbol{i})(v_1\mathbf{1} + v_2\boldsymbol{i}) \\ &= (u_1v_1 + u_2v_2)\mathbf{1} + (u_1v_2 - u_2v_1)\boldsymbol{i} \\ &= (u \cdot v)\mathbf{1} + (u \wedge v)\boldsymbol{i} \end{aligned}$$

very obviously involves both of them as its real and imaginary parts. The elements $\mathbf{1}$ and $\boldsymbol{i}$ in $\mathbb{C}$ play, in respect to the Hermitian product, the role of the basis $\overrightarrow{E}_1, \overrightarrow{E}_2$ in our story. Their key property is that the dot product and cross product are chosen to behave on them in the nicest possible way:

$$\overrightarrow{E}_1 \cdot \overrightarrow{E}_1 = \overrightarrow{E}_2 \cdot \overrightarrow{E}_2 = 1, \quad \overrightarrow{E}_1 \cdot \overrightarrow{E}_2 = 0 \tag{6.1}$$

and

$$\overrightarrow{E}_1 \wedge \overrightarrow{E}_1 = \overrightarrow{E}_2 \wedge \overrightarrow{E}_2 = 0, \quad \overrightarrow{E}_1 \wedge \overrightarrow{E}_2 = 1. \tag{6.2}$$

In the standard terminology, this basis is simultaneously *orthonormal* with respect to the dot product and *hyperbolic* with respect to the wedge product.

Reversely, the combination of Yin and Yang, of Euclidean and symplectic geometries, produces the geometry of complex numbers seen as a real plane: the magnificent new whole that is bigger than its constituent parts. And as soon as we have multiplication of complex numbers, we have a consistent treatment of rotations, and of inversion.


## *Acknowledgements*

I started to think about mathematics education systematically when in about 2012 Martin Hyland involved me in mathematics education policy discussions in the LMS, and in the Department for Education, and invited me to join CMEP, Cambridge Mathematics Education Project. The late Tony Gardiner was my guide in the murky world of education in Britain. Ali Nesin gave me a fantastic opportunity to teach, every summer from 2007 to 2019, experimental mathematics courses in his remarkable Mathematics Village in Turkey.


## *References*


**1.** M. Boersma. Domain-Specific Languages Made Easy. Manning Publication, 2020.

**2.** A. Borovik, *The Kolmogorov reform of mathematics education in the USSR*. In D. De Bock (Ed.), Modern Mathematics—An International Movement?, Springer International Publishing, pp. 319–336, 2023. Pre-publication version: arXiv:2210.03574 [math.HO].

**3.** A. Borovik and V. Kondratiev. A new course 'Algebra + Computer Science': What should be its outcomes and where it should start. Doklady Mathematics 107 Suppl. 1 (2023), S117–S131, 2023. Pre-publication version: arXiv:2212.12257v1 [math.HO].

**4.** A Borovik and Ş. Yalçınkaya. Adjoint representations of black box groups $PSL_2(F_q)$. J. Algebra 506, 540–591, 2018. Pre-publication version: arXiv:1502.06374 [math.GR].

**5.** H. S. M. Coxeter, Introduction to Geometry, 2nd ed. John Wiley & Sons, New York – London – Sydney – Toronto, 1969.

**6.** Dieudonné, J., Algèbre linéaire et géométrie élémentaire, Enseignement des Sciences, VIII, Hermann, Paris, 1964.

**7.** R. Hartshorne. Foundations of Projective Geometry, Lecture Notes, Harvard University, vol. 1966/67, W. A. Benjamin, Inc., New York, 1967.

**8.** M. Fowler. Domain Specific Languages. Addison-Wesley Professional, 2010

**9.** M. Voelter et al. DSL Engineering. Designing, Implementing and Using Domain-Specific Languages. 2010. `http://dslbook.org`.

**10.** V.V. Nikulin and I.R. Shafarevich. Geometry and Groups. Moscow, Nauka, 1983.



EMAIL `alexandre ≫at≪ borovik.net`
WEB www.borovik.net