Selected Passages from Correspondence with Friends 14 no. 3 (2026), 15–16 ISSN 2054–7145

WHY IS LINEAR ALGEBRA SO TEDIOUS?

ALEXANDRE BOROVIK

This is a basic law of economics of mathematics education: everything can be taught better, and without boredom, and mathematicians know how to do that — but it is more expensive.

There are two principal factors that create boredom in a linear algebra class:

(1) For variety of reasons, mostly for desire to make teaching more cost-efficient, Linerar Algebra is taught to students who are unprepared to take the course, in particular, who have no skills of abstract thinking, and no mastery of basic algebraic techniques. As a result, understanding of an elegant theory (which Linear Algebra is) is replaced by rote learning of dull (and, in many cases, badly formulated) computational routines.

(2) I taught Linear Algebra all my life, and I had never seen a textbook for FIrst Year students which would start with a careful set up of basic notions. Students get confused from day one. But a textbook that sets record straight and thus deviates from the established standards, is, in the present environment, unmarketable.

What I write further is an explanation of point (2) and is aimed more at a teacher than at a student.

It is well-known that the total cost of a purchase of amounts g_1, g_2, g_3 of some goods at prices p_1, p_2, p_3, respectively, is

$$p_1g_1 + p_2g_2 + p_3g_3 = \sum_{i=1}^{3} p_ig_i.$$

Expressions of this kind,

$$a_1x_1 + \cdots + a_nx_n$$

are called *linear forms* in variables $x_1, \ldots, x_n$ with coefficients $a_1, \ldots, a_n$

Linear Algebra studies the mathematics of linear forms. The linear form

$$p_1g_1 + p_2g_2 + p_3g_3$$

can be very conveniently written as

$$\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

2020 *Mathematics Subject Classification* 97G60.

and then abbreviated

$$\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \mathbf{p} \cdot \mathbf{g}$$

where $\mathbf{p}$ and $\mathbf{g}$ are vectors formed by their components

$$\mathbf{p} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$$

and

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

And at this point, the first didactical crime is committed in the standard textbooks: vectors $\mathbf{p}$ and $\mathbf{g}$ are treated as entities of the same kind. They are not; for example, they cannot be added: you do not add kilograms and euros. The difference between the two kinds of vectors involved in a linear form is emphasised by their different shapes. The use of dot product of vectors, say, in school level mechanics is misleading: students are told that work is the dot product of force and displacement, but they live in different vector spaces: you cannot add force and displacement.

So, the basic structure of Linear Algebra is a *pairing* of two vector spaces:

$$U \times V \longrightarrow \mathbb{R}.$$

This immediately calls for a more general definition of a vector space — because vector spaces involved could be of different nature — and systematic use of *duality* between vectors and *linear functionals*, between vectors and *covectors*.

Many years ago I taught a course in Linear Algebra in tensor notation from day one: in the example above, prices had upper indices p^j, while goods had lower indices g_i, and where a matrix element of a matrix A in row i and column j was a_i^j. It worked beautifully and allowed to make a seamless transition to tensors (I taught majors in Physics, it was essential) BUT, and it is an important BUT: my students had a much better preparation in basic algebra than we would normally expect now. On the plus side, the usual trouble with linear dependence / independence disappeared and compressed notation (which makes Linear Algebra so efficient) could be used with much greater freedom and confidence.

This short note is made of my answer to a question on Quora.

EMAIL `alexandre ≫at≪ borovik.net`
WEB www.borovik.net